Section 3.4

Rates of Change

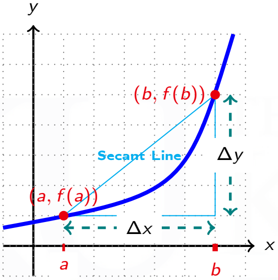
(1) 1-Unit Change

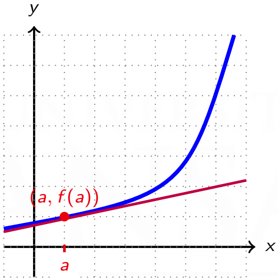
(2) Kinematics

(3) Gravity, Biology and Other Applications

For a function $y = f(x)$ over an interval $[a, b]$:

$$\Delta y = \text{ change in y } = f(b) - f(a) \qquad \Delta x = \text{ change in x } = b - a$$





Using this notation, the average rate of change on the interval is

$$\frac{\Delta y}{\Delta x} = \frac{f(b) - f(a)}{b - a}$$

and the instantaneous rate of change is

$$f'(a) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{b \to a} \frac{f(b) - f(a)}{b - a}$$

Example I: Let $A = \pi r^2$ be the area of a circle of radius r.

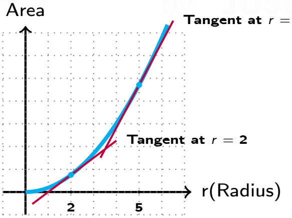
(A) Compute $\dfrac{dA}{dr}$ at $r = 2\,cm$ and $r = 5\,cm$.

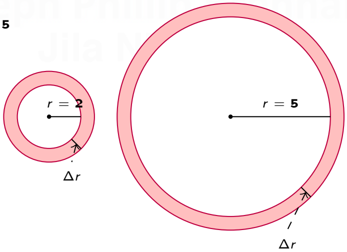
The rate of change of area with respect to radius is the derivative

$$\frac{dA}{dr} = \frac{d}{dr}\left(\pi r^2\right) = 2\pi r \ \frac{cm^2}{cm}$$

Therefore $\left.\dfrac{dA}{dr}\right|_{r=2} = 4\pi$ and $\left.\dfrac{dA}{dr}\right|_{r=5} = 10\pi$.

(B) Why is $\dfrac{dA}{dr}$ larger at $r = 5$?





The Effect of a 1-Unit Change

Assuming the derivative exists, for small values of h, the difference quotient is close to the derivative itself:

$$f'(a) \approx \frac{f(a+h) - f(a)}{h}$$

This approximation generally improves as h gets smaller, but in some applications, the approximation is already useful with $h = 1$:

$$f'(a) \approx \frac{f(a+1) - f(a)}{1} = f(a+1) - f(a)$$

Using $f'(a)$ to Estimate Change

The value of $f'(a)$ is an estimate of the change in $f(x)$ as x changes from a to $a+1$.

The Effect of a 1-Unit Change

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Example II: For speeds s between 30 mph and 75 mph, the stopping distance of a car after the brakes are applied is modeled by the function $F(s) = 1.1s + 0.05s^2$ ft. *Estimate the additional stopping distance required for a car traveling at 61 mph as opposed to 60 mph.*

Answer: $F'(s) = 1.1 + 0.1s$ ft/mph, so $F'(60) = 7.1$ ft/mph. Therefore, the change in stopping distance between 60 mph and 61 mph is approximately 7.1 ft.

(Using a calculator, $F(61) - F(60) = 7.15$ ft, so the estimate above is fairly accurate.)

Physics: Applications to Kinematics

Kinematics is the study of motion without consideration of mass or force.

Kinematics is all about calculus! (In fact, it was one of the original motivations for Newton to develop calculus as a separate branch of mathematics.)

Quantity	Symbol	Calculus	Units
Distance	$s(t)$		distance
Velocity	$v(t)$	$= s'(t)$	distance/time
Acceleration	$a(t)$	$= v'(t) = s''(t)$	distance/time^2
Jerk	$j(t)$	$= a'(t) = v''(t) = s'''(t)$	distance/time^3

We often use $h(t)$ (for height) instead of $s(t)$ when the motion is vertical.

Physics: Applications to Kinematics

In physics, polynomials are used to model how gravity affects the height of a projectile. Gravity on Earth provides a constant acceleration of -9.8 m/sec^2 $\approx$ -32 ft/sec^2.

By the power rule, the degree of the height function $h(t)$ is two higher than the degree of acceleration $a(t)$. Since acceleration is constant, it has degree zero, and it follows that the height polynomial is quadratic:

$$\begin{aligned} h(t) = pt^2 + qt + r \quad &\Rightarrow \quad v(t) = h'(t) = 2pt + q \\ &\Rightarrow \quad a(t) = v'(t) = h''(t) = 2p \end{aligned}$$

What do p, q, r signify? Since $a(t) = -9.8$ we have $p = -4.9$. Also, $q = v(0) = v_0$ is the initial velocity of the object, and $r = h(0) = h_0$ is the initial position.

$$\boxed{h(t) = -4.9t^2 + v_0 t + h_0}$$

Example III, Kinematics

A pineapple is thrown into the air. Its height (in feet) after t seconds is given by the function

$$h(t) = 24 + 40t - 16t^2.$$

(I) What are the initial height and velocity?

(II) When will the pineapple hit the ground?

(III) How fast is it going then?

(IV) What is the maximum height the pineapple reaches?

Example III, Kinematics

A pineapple is thrown into the air. Its height (in feet) after t seconds is given by the function

$$h(t) = 24 + 40t - 16t^2.$$

(I) The initial height is 24 ft and the initial velocity is 40 ft/sec (upward).

(II) The pineapple goes splat when $h(t) = 0$. Factorize: $h(t) = 8(1 + 2t)(3 - t)$, so $h = 0$ when $t = -1/2$ or $\boxed{t = 3.}$

(III) $h'(t) = 40 - 32t$ and so $h'(3) = 40 - 96 = \boxed{-56 \text{ ft/sec.}}$

(IV) The graph of $h(t)$ is a parabola, so the vertex (top point) will occur midway between the two zeroes, at $t = (-\frac{1}{2} + 3)/2 = \frac{5}{4}$ and $\boxed{h(\frac{5}{4}) = 49 \text{ ft.}}$

Physics: Gravity

Newton's Law of Gravitation says that the magnitude F of the force exerted by a body of mass m on a body of mass M is

$$F = \frac{GmM}{r^2}$$

where G is the gravitational constant $6.6741 \times 10^{-11}\ \mathrm{m}^3/\mathrm{kg} \cdot \mathrm{s}^2$ and r is the distance between the bodies.

We can calculate dF/dr, treating the masses as constants:

$$\frac{dF}{dr} = \frac{-2GmM}{r^3} = \frac{-2F}{r}$$

What is the physical interpretation of this?

— Gravitational force weakens at a slower rates as the distance between bodies increases.

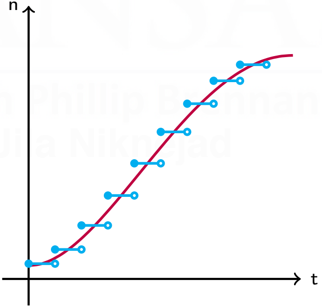
Biology: Population Growth

Let $n = f(t)$ be the number of individuals in an animal or plant population at time t. The average rate of population growth during the interval $[t_1, t_2]$ is

$$\frac{\Delta n}{\Delta t} = \frac{f(t_2) - f(t_1)}{t_2 - t_1}$$

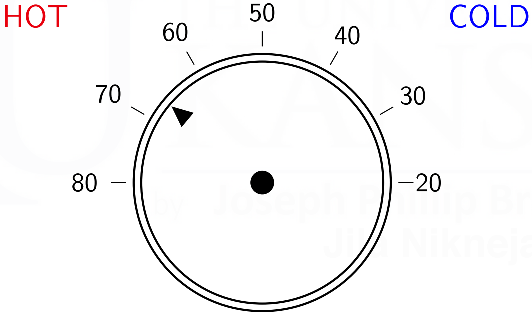
The **instantaneous** rate of growth is obtained by letting Δt approach zero. Strictly speaking, this is not accurate as population growth is **NOT** continuous — it is a step function.

For modeling purposes, we can approximate the graph of $n = f(t)$ by a smooth curve.



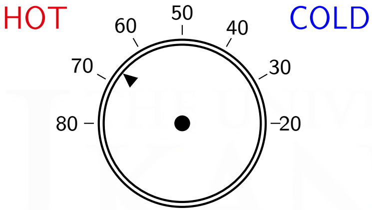
Calculus and Plumbing

The water temperature T in a shower (in $^\circ$C) is controlled by a knob. Let a be the angle of the knob in standard position (as shown).



What can we say about dT/da? **Units:** $^\circ$C / degrees of angle.

Calculus and Plumbing



Plumber's Troubleshooting Guide	
Problem	**Solution**
dT/da too small	Check the hot water supply
dT/da too large	Turn down the water heater
$dT/da = 0$	The knob isn't working — fix it
$dT/da < 0$	Knob is installed backwards!